Propagation of solitons in the Toda lattice with an impure segment

Yoji Kubota and Takashi Odagaki
Department of Physics, Kyushu University, Fukuoka 812-8581, Japan
(Received 26 July 1999)

The transmission and scattering of a single soliton is studied numerically in the Toda lattice with an impure segment which consists of two kinds of masses. The incident soliton is split into transmitted, reflected, and trapped solitons by the impure segment. The energy of the soliton trapped in the segment escapes from the segment very slowly and thus we can define the transmission rate by the ratio of energies of the transmitted soliton and the incident soliton. It is shown that the dependence of the transmission rate on the segment length N can be fitted quite well by $1/(1+\alpha N^{\beta})$. The transmission rate is also shown to be a monotone decreasing function of the wave number of the incident soliton. Most of the energy of the transmitted wave is carried by a large soliton (the frontier soliton) at the front, which is shown to be an exact soliton of the Toda lattice. When the mass difference is small, the transmission rate can be obtained by considering the segment as a repetition of a unit and repeating the renormalization of the wave number due to the unit.

PACS number(s): 42.81.Dp, 05.45.Yv

I. INTRODUCTION

Propagation of solitons in inhomogeneous media shows many important aspects such as deformation and disruption of solitons, excitation of localized modes, and scattering of solitons [1–14]. In particular, the generation of localized modes and scattering of incident solitons due to impurities are important problems in the study of the basic properties of soliton propagation and in practical applications. It is well known that a harmonic chain with a light mass impurity has a stable localized mode whose amplitude decays exponentially with the distance from the impurity [15]. Many numerical studies have shown that there are localized modes in nonlinear media with impurities. The localized mode due to a single impurity in the Toda lattice behaves similar to the localized state in a harmonic chain with a single impurity [1,7,8]. When nonlinearity and inhomogeneity are weak enough, the scattering problem of solitons can be treated analytically by a perturbation method. In this case, the deviation from the one soliton state is insignificant. For example, the Toda lattice with a single mass impurity has been studied with the perturbation method on the basis of the inverse scattering transformation by Yajima [5]. If the difference of mass ϵ is small enough, the decrease δA of the transmitted soliton amplitude was shown to behave as $\delta A \propto (\epsilon A)^2$, where A is the amplitude of the incident wave. Transmission of a soliton in random media with quartic nonlinearity has been studied theoretically and numerically [10,12,14]. The time dependence of the amplitude of the transmitted soliton was shown to behave as $t^{-1/2}$ for large t in the continuum limit [10]. When nonlinearity and inhomogeneity are not weak, it is virtually impossible to obtain the solution analytically.

In this paper, we investigate numerically fundamental aspects of soliton scattering due to impure segments. In the present study, we treat the Toda lattice as a representative system which supports solitons. It should be emphasized that the treatment presented here will give some basic principles in the study of soliton scattering and transmission in general. The reason that we treat the Toda lattice is that (1) since the exact solution exists for the Toda lattice without continuum approximation, we can change the wave number of the soliton as large as we want and (2) since there exists a nonlinear LC circuit that is identical to the Toda lattice as suggested by Hirota and Suzuki [11], we can test our numerical results by experiments. The basic study for the soliton propagation in media subject to random modifications is also relevant and important in the optical soliton communication in fiber optics [16].

Our model is introduced in Sec. II, where we define basic quantities which are used in the following discussion. When a soliton is launched into a segment with random mass distribution, we observe transmitted, reflected, and trapped solitons. The trapped soliton appears to be localized in the segment and to gradually escape from the segment. We call the trapped soliton a quasilocalized soliton. We define quantities describing soliton propagation in this situation. In Sec. III, we present the numerical results when a soliton is launched into a finite impure segment. We study the dependence of the scattering characteristics on the length of the segment and the wave number of the soliton. We will see that the transmitted wave consists of many peaks, the first one of which is an exact soliton. In Sec. IV, we discuss the time dependence of the trapped solitons in the impure segment. We give a simple analysis of the scattering process in Sec. V. Section VI is devoted to discussion.

II. MODEL

We consider the Toda lattice [17] whose Hamiltonian is given by

$$H=\sum_n \left[\frac{1}{2m_n}p_n^2+\frac{a}{b}(e^{-bu_n}+bu_n-1)\right], \tag{2.1}$$

$$u_n=q_{n+1}-q_n, \tag{2.2}$$

where m_n is the mass of a particle at site n, and p_n and q_n are the momentum and displacement of the particle on site n, respectively. The interaction between particles on sites n and

FIG. 1. The perspective of an incident soliton on the Toda lattice with the impure segment $1 \leqslant n \leqslant N$.

$n+1$ is determined by a and b, where b is a controlling parameter of the nonlinearity. In fact, if we take the limit of $b=0$ keeping $K \equiv ab$ constant, the potential energy becomes the harmonic form $\Sigma_n \frac{1}{2} K(q_{n+1}-q_n)^2$. We embed a finite segment of impure masses in a homogeneous infinite chain. Namely, we set

$$m_n = \begin{cases} m & \text{for } n \leqslant 0 \text{ and } n \geqslant N+1 \\ m(1+\epsilon\gamma_n) & \text{for } 1 \leqslant n \leqslant N \end{cases}, \quad (2.3)$$

where γ_n is assumed to be 1 or -1 and $\epsilon \in (0,1)$ denotes the difference of two kinds of masses. We place equal number ($N/2$) of $\gamma_n = 1$ and $\gamma_n = -1$ randomly in the segment of N sites, so that the average mass of the segment is the same as the regular part.

In order to perform the numerical study, we introduce dimensionless variables for time, momentum, and displacement which are defined by

$$\tau = \Omega t, \quad (2.4)$$

$$P_n = \frac{b}{m\Omega} p_n, \quad (2.5)$$

$$Q_n = b q_n, \quad (2.6)$$

where $\Omega = \sqrt{ab/m}$. The equations of motion reduce to

$$\begin{cases} \dfrac{\partial}{\partial \tau} Q_n = \dfrac{P_n}{1+\epsilon\gamma_n} \\ \dfrac{\partial}{\partial \tau} P_n = -(e^{-U_n}-1) + (e^{-U_{n-1}}-1), \end{cases} \quad (2.7)$$

where $U_n = Q_{n+1} - Q_n$. At time $\tau = 0$, we prepare one soliton solution

$$U_n = -\ln[1+\omega_0^2 \cosh^{-2}(k_0 n - \omega_0 \tau)]|_{\tau=0}, \quad (2.8)$$

$$\omega_0 = \sinh k_0 \quad (2.9)$$

at the far left of the segment and let it propagate to the right (Fig. 1). The energy of the soliton in the unit of a/b is given by

$$E_0 = 2(\sinh k_0 \cosh k_0 - k_0). \quad (2.10)$$

Here, k_0 is regarded as the wave number of the soliton scaled by b.

When the soliton passes the impure segment, we observe three waves subsequently: transmitted (T), reflected (R), and quasilocalized (L) waves (Fig. 2). The energy of the incident soliton is distributed into these three waves. In order

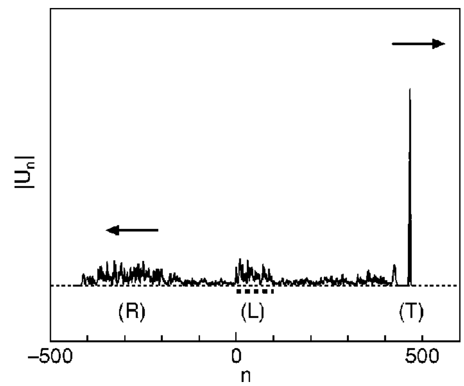

FIG. 2. The profile of waves, after some time the soliton passes the impure segment at [1,100] (denoted by a thick dashed line) which consists of transmitted (T), reflected (R), and quasilocalized (L) waves. U_n is relative displacement scaled by $1/b$.

to study the scattering process, we define three temporal quantities which represent the fraction of the incident energy in these three waves:

$$T(\tau) = E_T(\tau)/E_0, \quad (2.11)$$

$$R(\tau) = E_R(\tau)/E_0, \quad (2.12)$$

$$L(\tau) = E_L(\tau)/E_0, \quad (2.13)$$

where $E_T(\tau)$ and $E_R(\tau)$ are the energy in the regions left and right of the segment, respectively, and $E_L(\tau)$ is the energy in the segment. As we can see from the following numerical results, the transmitted soliton consists of a large soliton at the front and many small waves following it, and most of the energy is carried by the large soliton. We call the large soliton the frontier soliton. Therefore we can use the frontier soliton transmission coefficient

$$T_1 = E_1/E_0 \quad (2.14)$$

as the measure of the transmission coefficient [13,14], where E_1 is the total energy of the frontier soliton. In Ref. [14], it is called ''the first soliton transmission coefficient.'' We use these quantities to analyze the scattering process of a soliton in the following sections.

III. TRANSMISSION PROFILES

We prepare a chain of 500 sites in which we embed an impure segment of 100 sites at [1,100]. We integrate Eq. (2.7) numerically using the fourth-order Runge-Kutta method and obtain the spatiotemporal evolution of energy density $h_n(\tau)$ which is given by

$$h_n(\tau) = \frac{1}{2E_0} \left\{ \frac{P_n^2(\tau)}{1+\epsilon\gamma_n} + [e^{-U_n(\tau)} + U_n(\tau) - 1] + [e^{-U_{n-1}(\tau)} + U_{n-1}(\tau) - 1] \right\}. \quad (3.1)$$

We note that $h_n(\tau)$ is normalized by the total energy of incident soliton. Figure 3 shows $h_n(\tau)$ as a function of n and τ when the difference of two kinds of masses and the wave

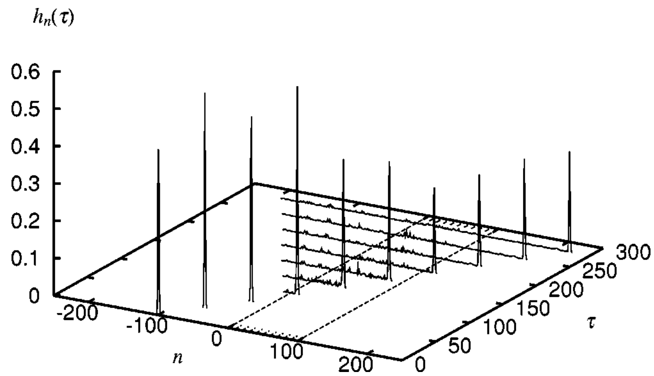

FIG. 3. The spatiotemporal evolution of energy density $h_n(\tau)$ which is normalized by total energy. τ is dimensionless time defined by $\tau=\sqrt{ab/m}t$.

number of the incident soliton are $\epsilon=0.2$ and $k_0=1.0$. The incident soliton is scattered by the segment, and transmitted and reflected waves are produced. At the same time a part of the incident soliton energy is captured in the impure segment. The characteristics of the transmitted wave depend on the strength of the impure segment and the amplitude of the incident soliton. We observed that when the mass difference ϵ is small and the amplitude of the incident soliton is small, the transmitted wave keeps its shape as soliton and propagates without much effect. When the irregularity is strong, the transmitted wave is disintegrated into a frontier soliton and many little waves which follow the frontier soliton. In Fig. 4, we show $T(\tau)$, $R(\tau)$, and $L(\tau)$ as functions of time where $E_T(\tau)=\Sigma_{n=101}h_n(\tau)$,$E_R(\tau)=\Sigma_{n=1}^{100}h_n(\tau)$, and $E_L(\tau)=\Sigma_{n=-200}^{0}$ are utilized in Eqs. (2.11)–(2.13). Since our model does not have any energy dissipation, the total energy should be conserved, namely,

$$T(\tau)+R(\tau)+L(\tau)=1. \tag{3.2}$$

This sum rule was confirmed to hold in our results within the discretization error.

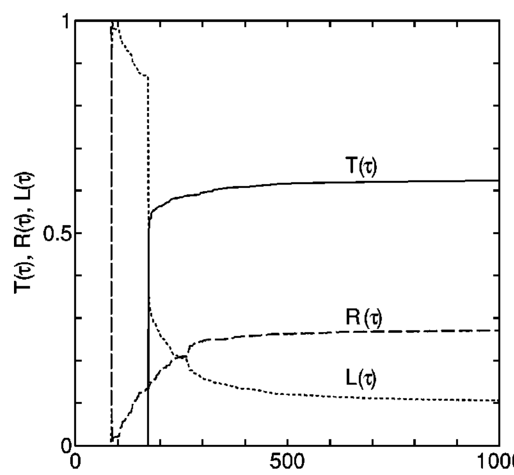

FIG. 4. The energy of transmitted $T(\tau)$, reflected $R(\tau)$, and localized $L(\tau)$ solitons as functions of time. τ is dimensionless time defined by $\tau=\sqrt{ab/m}\,t$. The impure segment length, the wave number of the incident soliton, and the difference of two kinds of masses are $N=100$, $k_0=1.0$, and $\epsilon=0.2$, respectively.

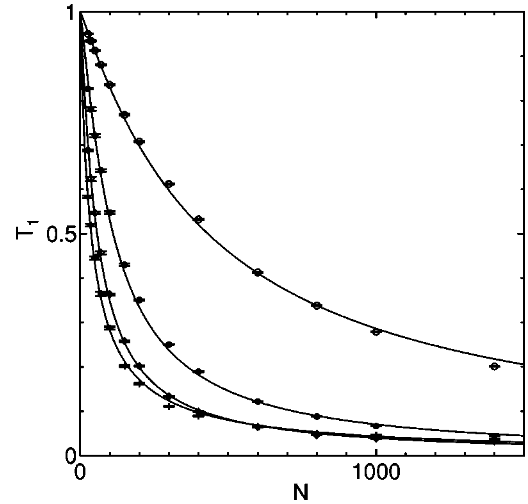

FIG. 5. The frontier soliton transmission coefficient T_1 is plotted as a function of the segment length N with error bars for $\epsilon=0.1$ (○), 0.2 (●), 0.3 (◇), and 0.4 (+). The wave number k_0 of the incident soliton is 1.0. The solid curves are the fit by Eq. (3.3).

In Fig. 4, the abrupt decrease in $R(\tau)$ accompanied with the abrupt increase in $L(\tau)$ indicates that the soliton entered the segment at $\tau=85$. Similarly, the sudden increase in $T(\tau)$ near $\tau=175$ denotes that a part of the incident soliton is leaving the segment. The energy in the impure segment $L(\tau)$ decreases very slowly after some time and then $T(\tau)$ and $R(\tau)$ are regarded as constants which can be used to define the transmission and reflection coefficients.

Since most part of the energy in the transmitted waves is carried by the frontier soliton, we use mostly the frontier soliton transmission coefficient T_1 in the following discussion.

A. *N* dependence of T_1

We obtained T_1 for various length N of the impure segment keeping other parameters fixed. The total energy E_0 and the wave number k_0 of the incident soliton are 1.6269 and 1.0. Figure 5 shows the N dependence of T_1 for $\epsilon=0.1$, 0.2, 0.3, and 0.4, which were obtained by the average over 20 samples.

The N dependence of T_1 can well be fitted by

$$T_1(N)\simeq\frac{1}{1+\alpha N^{\beta}}. \tag{3.3}$$

The values of α and β are summarized in Table I. Therefore,

TABLE I. The constants α and β in function $1/(1+\alpha N^{\beta})$, which represents N dependence of T_1 for the difference of two kinds of masses $\epsilon=0.1$, 0.2, 0.3, and 0.4. The wave number of incident soliton is fixed in $k_0=1.0$.

ϵ	α	β
0.1	0.001376	1.086
0.2	0.003952	1.174
0.3	0.01114	1.112
0.4	0.03384	0.9373

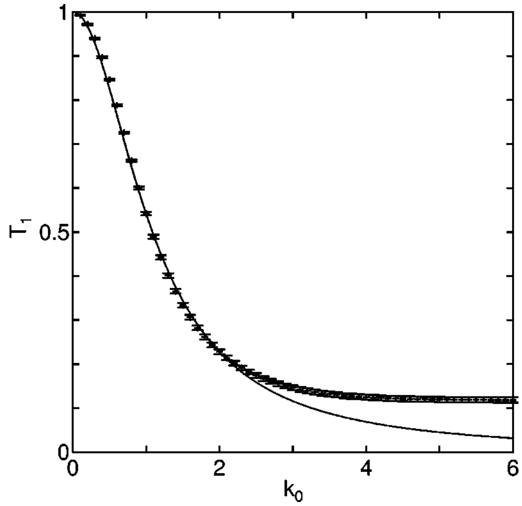

FIG. 6. The frontier soliton transmission coefficient $T_1(+)$ as a function of the wave number k_0 of the incident soliton with error bars. The wave number k is scaled by b. The impure segment length and the difference of two kinds of masses are $N=100$ and $\epsilon=0.2$. The solid curve denotes the Lorentzian function (3.4).

for large N we have $T_1 \sim N^{-\beta}$.

B. k_0 dependence of T_1

Keeping $N=100$ and $\epsilon=0.2$ fixed, we calculated T_1 for solitons with various wave numbers. In Fig. 6, T_1's are plotted against the wave number k_0 of the incident soliton, which were calculated from the average over 10 samples. We can see from Fig. 6 that T_1 decreases with k_0. In the small k_0 region, $k_0 \lesssim 2.2$, the decay of T_1 is represented by a Lorentzian function

$$T_1 \simeq \frac{c}{k_0^2+c} \tag{3.4}$$

with $c=1.178$. In the large k_0 limit, T_1 tends to a finite value.

C. Frontier soliton

The frontier soliton seems to propagate without decay after it entered into the homogeneous region on the right hand side. To confirm this, we show in Fig. 7 the comparison between the frontier soliton and an exact soliton with the same energy. Agreement of the shape of these solitons is excellent, and thus it is concluded that the scattering process generates a new soliton.

IV. QUASILOCALIZED SOLITONS

A part of the incident soliton is trapped in the impure segment and the trapped energy escapes from the segment gradually. Figure 8 shows the decay of the trapped energy for $\epsilon=0.1$, 0.2, 0.3, and 0.4 in the logarithmic scale. The segment length and the wave number of the incident soliton are $N=100$ and $k_0=1.0$.

In Fig. 8, the initial weak decay for $\tau \lesssim 170$ represents the decay due to the reflection, and the following rapid decrease is produced by the escape of the frontier soliton from the segment. In the time region, $180 \lesssim \tau \lesssim 500$, the decay of $L(\tau)$ can be approximated by $\sim e^{\xi/\tau}$, the value of parameter ξ is summarized in Table II. The decay in the long time region can be fitted by a power-law function $\sim \tau^{-\mu}$ whose exponent is also summarized in Table II. The crossover indicates that stable trapped modes are found in the segment and that the trapped modes escape from the region at a much slower rate.

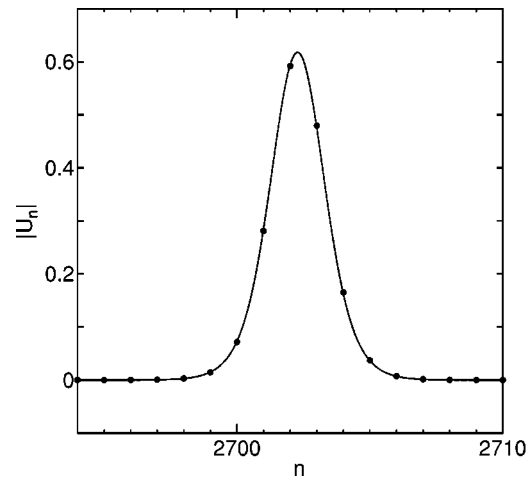

FIG. 7. Comparison of the frontier soliton (●) and the exact soliton of the Toda lattice (—) with the same energy. U_n is relative displacement scaled by $1/b$.

V. SIMPLE ANALYSIS

We analyze the N dependence of the frontier soliton transmission coefficient from the results obtained in Sec. III. We plot the wave number k_t of the frontier soliton transmitted from an impure segment against the wave number k_i of the

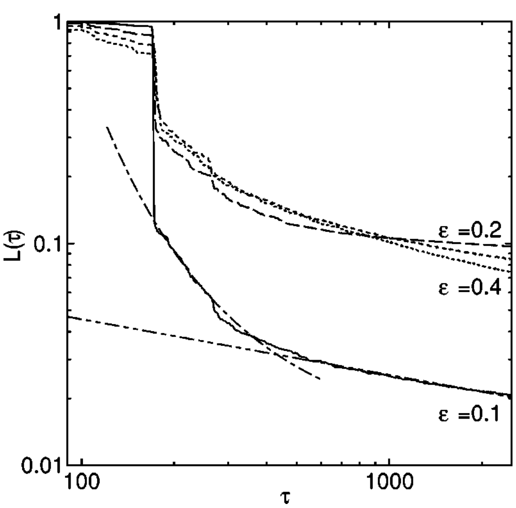

FIG. 8. The energy $L(\tau)$ of the solitons trapped in the impure segment is plotted as a function of time in the logarithmic scale for the difference of two kinds of masses $\epsilon=0.1$ (the solid line), 0.2 (the dashed line), 0.3 (the short dashed line), and 0.4 (the dotted line). τ is dimensionless time defined by $\tau=\sqrt{ab/m}\,t$. The impure segment length and the wave number of the incident soliton are $N=100$ and $k_0=1.0$. The dot-dashed curve denotes fitting function $\sim e^{\xi/\tau}$ and the two-dot–dashed line denotes $\sim \tau^{-\mu}$ for $\epsilon=0.1$.

TABLE II. The coefficient of functions which represents time dependence of the energy $L(\tau)$ of the solitons localized in the impure segment for the difference of two kinds of masses $\epsilon=0.1,\ 0.2,\ 0.3$, and 0.4. The length of the impure segment and the wave number of the incident soliton are $N=100$ and $k_0=1.0$, respectively. In the intermediate time region $L(\tau)\sim e^{\xi/\tau}$ and in the long time region $L(\tau)\sim\tau^{-\mu}$.

ϵ	ξ	μ
0.1	397.1	0.2511
0.2	272.4	0.1073
0.3	290.6	0.2898
0.4	270.4	0.3694

incident soliton, where k_t is determined numerically from T_1 by

$$E_iT_1=2(\sinh k_t\cosh k_t-k_t), \tag{5.1}$$

where E_i is the total energy of the incident soliton. Figure 9 shows k_t as a function of k_i for the impure segment length and the mass difference are $N=100$ and $\epsilon=0.2$.

We have shown that the frontier transmitted soliton is the exact soliton of the Toda lattice in Sec. III. Suppose a soliton propagates through a long impure segment of $N=nN_0$ sites. (As an example, we set $N_0=100$.) Then, we can consider that the soliton goes through a segment of N_0 sites one by one and each segment changes the wave number of the soliton according to the relation shown in Fig. 9. Thus by repeating the renormalization of the wave number $k_i\mapsto k_t$ n times, we can estimate the N dependence of T_1 from Eq. (5.1) which is plotted in Fig. 10 together with the result obtained in Sec. III. We can see from Fig. 10 that when ϵ is small ($\lesssim 0.2$) the renormalization method works quite well, and the agreement becomes poor for large ϵ ($\gtrsim 0.3$). The discrepancy for large ϵ is due to the fact that the soliton is reflected at the boundary between successive segments.

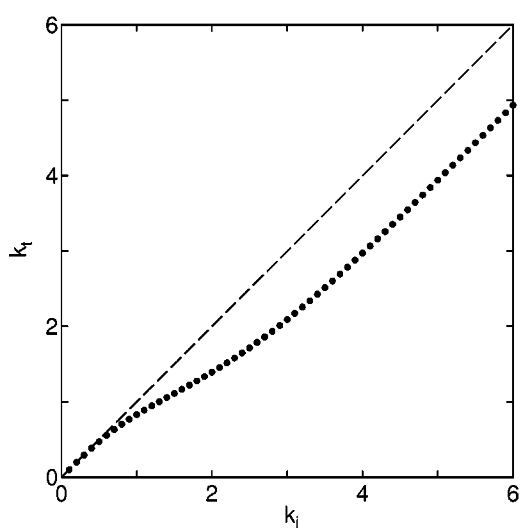

FIG. 9. The wave number k_t (●) of the transmitted frontier soliton as a function of the wave number k_i of the incident soliton. The wave number k is scaled by b. The impure segment length and the difference of two kinds of masses are $N=100$ and $\epsilon=0.2$. The dashed line denotes $k_t=k_i$.

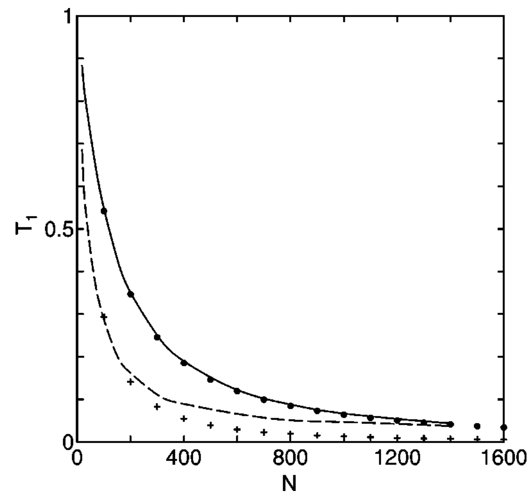

FIG. 10. The N dependence of the frontier soliton transmission coefficient T_1 obtained by the renormalization of the wave number at every 100 sites for the difference of two kinds of masses $\epsilon=0.2$ (●) and 0.4 (+). N is the total length of the impure segment. The wave number of the initial incident soliton is $k_i=1.0$. The solid and dashed lines denote the N dependence obtained in Sec. III for $\epsilon=0.2$ and $\epsilon=0.4$, respectively.

VI. SUMMARY

We have studied numerically the propagation of solitons through a segment with impure masses and investigated the characteristic feature in the scattering process of solitons by the segment. The incident soliton propagates emitting localized modes due to the breaking of integrability by the random mass segment. The localized modes radiate ripples, and localized energy escapes gradually from the impure segment. We defined the transmission, reflection, and quasilocalized rate using the distribution of the incident energy in these waves. We also defined the frontier soliton transmission coefficient T_1 and showed that T_1 behaves as $1/(1+\alpha N^{\beta})$ as a function of length N of the impure segment. We also showed that T_1 depends on the energy or the wave number of the incident soliton. It is interesting to note that the frontier transmitted soliton is also an exact soliton of the Toda lattice. This indicates that we can control the soliton transmission by impurities. For example, it will be possible to construct a soliton filter which prevents a selected soliton from passing through the segment.

In the present numerical study, we used one soliton as the incident wave. It will also be interesting to study the transmission of many incident solitons. In this case, the frontier soliton transmission coefficient T_1 will no longer be well defined and we must take account of the energy distribution in all solitons. In this situation several questions arise: Is it possible to realize a stationary state by many solitons? What is the interaction between solitons and localized oscillations? How is the interval between two solitons affected by reflected waves? These are subjects to be studied in the future.

ACKNOWLEDGMENT

This work was supported in part by Grants-in-Aid from the Ministry of Education, Science, and Culture.

[1] F. Yoshida, T. Nakayama, and T. Sakuma, J. Phys. Soc. Jpn. **40**, 901 (1976).
[2] F. Yoshida and T. Sakuma, Prog. Theor. Phys. **60**, 338 (1978).
[3] M. Aoki and Y. Tabata, J. Phys. Soc. Jpn. **54**, 23 (1985).
[4] A. Nakamura, Prog. Theor. Phys. **59**, 1447 (1978).
[5] N. Yajima, Phys. Scr. **20**, 431 (1979).
[6] H. Flaschka, Prog. Theor. Phys. **51**, 703 (1974).
[7] S. Watanabe and M. Toda, J. Phys. Soc. Jpn. **50**, 3436 (1981).
[8] S. Watanabe and M. Toda, J. Phys. Soc. Jpn. **50**, 3443 (1981).
[9] K. Nagahama and N. Yajima, J. Phys. Soc. Jpn. **58**, 1539 (1989).
[10] T. Iizuka, T. Nakao, and M. Wadati, J. Phys. Soc. Jpn. **60**, 4167 (1991).
[11] R. Hirota and K. Suzuki, J. Phys. Soc. Jpn. **28**, 1366 (1970).
[12] G.S. Zavt, M. Wagner, and A. Lütze, Phys. Rev. E **47**, 4108 (1993).
[13] Q. Li, St. Pnevmatikos, E.N. Economou, and C.M. Soukoulis, Phys. Rev. B **37**, 3534 (1988).
[14] Q. Li, C.M. Soukoulis, St. Pnevmatikos, and E.N. Economou, Phys. Rev. B **38**, 11 888 (1988).
[15] K. Ishii, Prog. Theor. Phys. Suppl. **53**, 77 (1973).
[16] Govind P. Agrawal, *Nonlinear Fiber Optics*, 2nd ed. (Academic Press, New York, 1995).
[17] M. Toda, J. Phys. Soc. Jpn. **22**, 431 (1967).